Invariant power law distribution of Langevin systems with colored multiplicative noise

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The random multiplicative process is studied for the case of a colored multiplicative noise with exponentially decreasing autocorrelation function. We observe the power law exponent of probability distribution in a statistically steady state numerically to clarify the effect of finite correlation time. The renormalization procedure is applied to derive the power law exponent theoretically. The power law exponent is inversely proportional to the autocorrelation time of the multiplicative noise.

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I. INTRODUCTION

Fluctuations following power law distributions have been found in various fields, such as size distribution of avalanches in a sandpile model [1], probability distribution of stock market price changes in economics [2], and probability distribution of the fluctuations of the intervals of heartbeats [3]. There have been many attempts to clarify the mechanism of the power law behavior. For example, the concept of self-organized criticality is based on the idea that open systems showing the power law exhibited by some open systems may be understood in analogy to automatic setting of the parameters at the critical point of a second-order phase transition [1].

The random multiplicative process (RMP), which is a generalized version of the Langevin equation involving a randomly multiplicative coefficient, is another type of mechanism generating a power law behavior. It has been widely introduced as a model to understand aspects of the singular behavior in nonlinear dynamics, such as the on-off intermittency [4,5], conformation of polymers in random velocity field [6], the model of stock market price changes in economics [7], and so on [8,9].

In our previous study [10] we have proved rigorously that the discrete version of the RMP with both a positive multiplicative and additive noises converges to a statistically steady state that is characterized by power law distribution. We have analytically obtained the equation for determining the power law exponent as a function of a stochastic property of multiplicative noise by using a characteristic function method. The exponent β is given by the value where the β-order moment of the multiplicative noise is equal to unity. The existence of the additive noise is essential but its statistical properties are known to be less relevant. It has been analytically proved that the probability density of the RMP follows power law [4,11,12].

It is useful to consider the effect of correlated multiplicative noise to extend the applicability of the RMP to complicated systems in the real world. For example, aspects of the behavior in on-off intermittency [4,5] and the noisy coupled maps [13] can be regarded as RMPs with colored multiplicative noise, and they have been analyzed by using the *local* or *finite time Lyapunov exponent*. In the study of particles' motion in turbulence Deutsch has been interested in the RMP with colored multiplicative noise [14] since the velocity field in turbulence has strong autocorrelation. Čenys and Lustfeld have studied the case of the RMP with chaotic multiplicative noise in on-off intermittency [5] and Nakao [16] has reported that the RMP with colored multiplicative noises shows the power law behavior by numerical simulations.

A question naturally arises whether there is a simple relation between the autocorrelation function of multiplicative noise and the power law exponent as well as in the case of white multiplicative noise. The aim of this paper is to quantitatively investigate the power law behavior of the RMP with colored multiplicative noise by numerical simulations and to clarify the relation between the autocorrelation and the power law exponent theoretically.

In Sec. II we perform a numerical simulation of the RMP and observe the relation between the exponent of the power law distribution and the autocorrelation time. In Sec. II based on the idea of renormalization procedure we analytically estimate the relation. We discuss the validity of the method that we used in the theoretical approach in Sec. IV and concluding remarks are summarized in Sec. V.

II. NUMERICAL SIMULATIONS

A. Model and simulation

We treat the discrete version of the RMP

$$x(t+1)=b(t)x(t)+f(t), \tag{1}$$

where multiplicative noise $b(t)$ represents random dissipation ($b<1$) or magnification ($b>1$) and $f(t)$ represents an additive noise. In the following analysis we assume that $b(t)$ is statistically independent of $f(t)$, so that the cross correlation between $b(t)$ and $f(t)$ vanishes, and we use the additive noise $f(t)$ that is a zero-mean white Gaussian given by

$$U(f)=\frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{f^2}{2\sigma^2}\right), \tag{2}$$

where the standard deviation is fixed as $\sigma=0.01$ throughout our numerical simulation.

For simplicity we specify the colored multiplicative noise as a noise having the following stationary, Gaussian, Markovian property that has zero average and variance V:

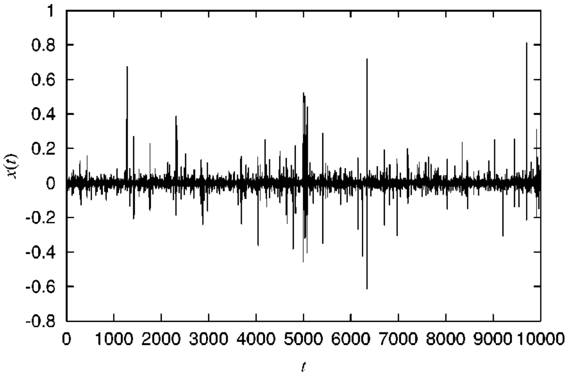

FIG. 1. Time series of $x(t)$ for $V=1.0$ and $\tau_c=1.0$ in the case of the colored Gaussian multiplicative noise.

$$W(b)=\frac{1}{\sqrt{2\pi V}}\exp\left(-\frac{b^2}{2V}\right) \tag{3}$$

with autocorrelation function given by

$$R(\tau)=\langle b(t)b(t+\tau)\rangle-\langle b(t)\rangle\langle b(t+\tau)\rangle, \tag{4}$$

which decays exponentially

$$R(\tau)=V\exp(-\tau/\tau_c), \tag{5}$$

where τ_c is an autocorrelation time. It is known that we can easily generate such types of random sequences by an autoregressive stochastic process [15]. In order to satisfy the stochastic properties Eqs. (3) and (5), we take the following stochastic process:

$$e^{1/\tau_c}b(t+1)-b(t)=V\sqrt{e^{2/\tau_c}-1}\,w(t), \tag{6}$$

where $w(t)$ is a normalized zero-mean white Gaussian noise.

We make numerical simulations of Eq. (1) for parameters V and τ_c. We show an example of time series $x(t)$ in Fig. 1. For investigating the tails in the probability distribution it is convenient to use the cumulative distribution function (CDF), $P(\geqslant|x|)$, defined as

$$P(\geqslant|x|)=\int_{-\infty}^{-|x|}p(x')dx'+\int_{|x|}^{\infty}p(x')dx', \tag{7}$$

where $p(x)$ denotes a probability density function of x. When the probability density has the power law tails

$$p(x)\propto|x|^{-\beta-1}\quad(0<\beta<2), \tag{8}$$

one has

$$P(\geqslant|x|)\propto|x|^{-\beta}, \tag{9}$$

and we can estimate the power law exponent β from its slope of the log-log plot. We also take the CDF from a time series on 10^6 steps for each set of V and τ_c. As shown in Fig. 2 we can find clear power law tails in the CDFs. In all cases we can find power law distributions and the exponent β obviously depends on both V and τ_c.

In order to investigate more quantitatively the dependence of β on V and τ_c we plot β as a function of V and τ_c. From Fig. 3 we may expect that the form of $\beta(V,\tau_c)$ is simple since we find isometric β curves mutually nonintersecting. Especially noticing the relation between β and τ_c, from Fig. 4, which is taken for fixed V, β is estimated to be inversely proportional to τ_c for large value of τ_c, namely, we have

$$\beta\propto\tau_c^{-1}. \tag{10}$$

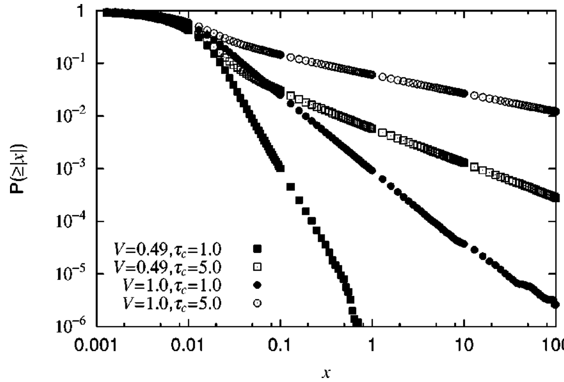

FIG. 2. Log-log plots of the cumulative distributions of x for various values of V and τ_c. Filled squares represent $V=0.49$ and $\tau_c=1.0$, unfilled squares $V=0.49$ and $\tau_c=5.0$, filled circles $V=1.0$ and $\tau_c=1.0$, and unfilled circles $V=1.0$ and $\tau_c=5.0$.

B. The white Gaussian limit

The colored Gaussian noise given by Eq. (6) converges to a white Gaussian noise in the limit of $\tau_c\to 0$. Here, we discuss the limit of the zero-mean white Gaussian multiplicative noise. For this case we cannot directly apply the formula obtained in our previous study to the zero-mean white Gaussian noise in order to estimate the power law exponent β because they take both positive and negative values. In the same way as we have shown in our previous study we can easily generalize the formula for a noise having both positive and negative values

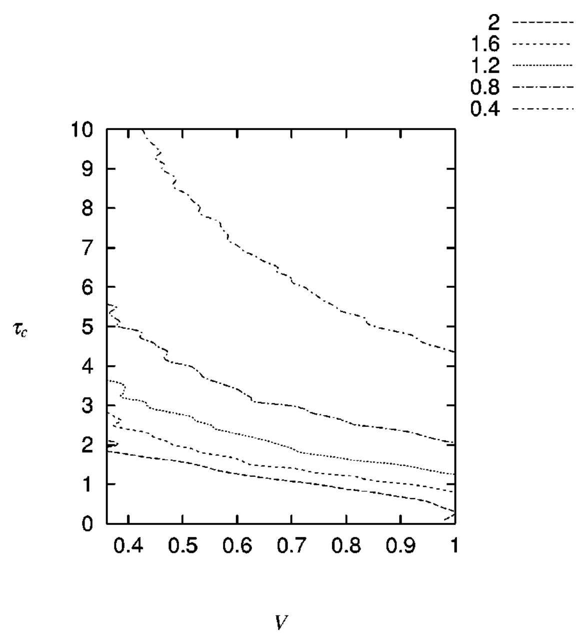

FIG. 3. Each curve represents isometric β for the function $\beta(V,\tau_c)$.

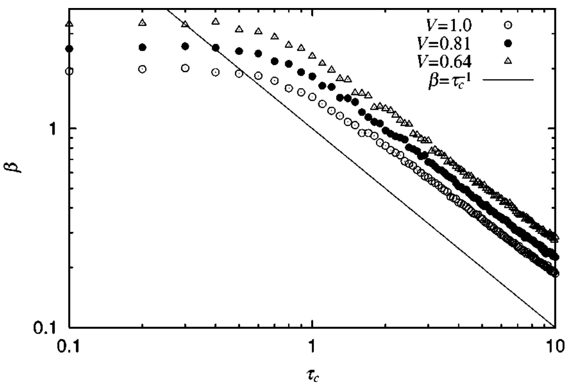

FIG. 4. Log-log plots of relation between β and τ_c on some fixed values of V. Each mark represents a numerical result for V, and a solid line represents $\beta=\tau_c^{-1}$.

$$\langle |b|^{\beta}\rangle = 1 \quad (0<\beta<2). \tag{11}$$

Using Eq. (11) one can analytically calculate β as a function of a variance V of the Gaussian for the case of zero-mean white Gaussian multiplicative noise. Applying Eq. (3) to Eq. (11) we have

$$\frac{1}{\sqrt{\pi}}(2V)^{\beta/2}\Gamma\left(\frac{\beta+1}{2}\right)=1, \tag{12}$$

where $\Gamma(\cdots)$ is the gamma function.

We show an example of time series in Fig. 5 in the white Gaussian limit. We take the CDF from 10^6 steps of the time series for each fixed value of V. It is found that for each value of V a slope of log-log plot changes in Fig. 6. We obtain the relation between V and β from each slope of the CDF. As shown in Fig. 7 we can confirm that the theoretical solution Eq. (12) fits the numerical estimation precisely. These results mean that one can fully characterize the power law exponent as a feature of multiplicative noise in the limit $\tau_c \to 0$. Therefore, we find that β converges to a finite value on the limit of $\tau_c \to 0$ in spite of Eq. (10) as shown in Fig. 4.

C. Intuitive understanding of the result

We can give the intuitive reason why the power law exponent β is inversely proportional to τ_c in the following way. We may deal with b^{τ_c} as an independent set of multiplicative noises because we can roughly estimate that multiplicative noises keep the same values in the interval τ_c. Thus we may apply Eq. (11) for a white multiplicative noise to a renormalized multiplicative noise b^{τ_c},

$$\langle |b^{\tau_c}|^{\beta}\rangle = 1. \tag{13}$$

Since the renormalized multiplicative noise may be expected to depend on V alone, we find the exponent $c(V)$ for $\beta\tau_c$,

$$\beta\tau_c = c(V), \tag{14}$$

where $c(V)$ is a power law exponent for the renormalized multiplicative noise b^{τ_c} and a function of V alone. Thus β is inversely proportional to τ_c. In Sec. III we study the relation between the power law exponent and the autocorrelation time more rigorously based on an analytical estimation of the intuitive idea of renormalization.

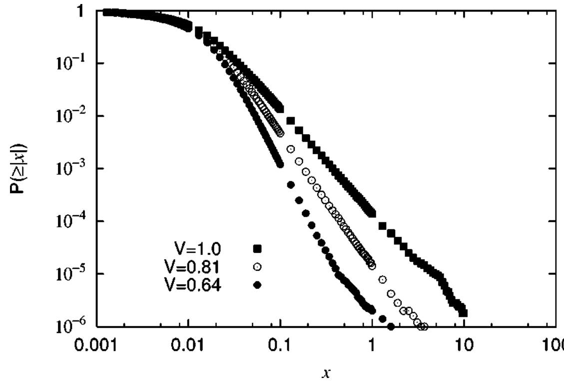

FIG. 6. Log-log plots of the cumulative distributions of x for $V=1.0$, $V=0.81$, and $V=0.64$ in the case of the white Gaussian multiplicative noise. We find that each slope of plots changes for the value of V.

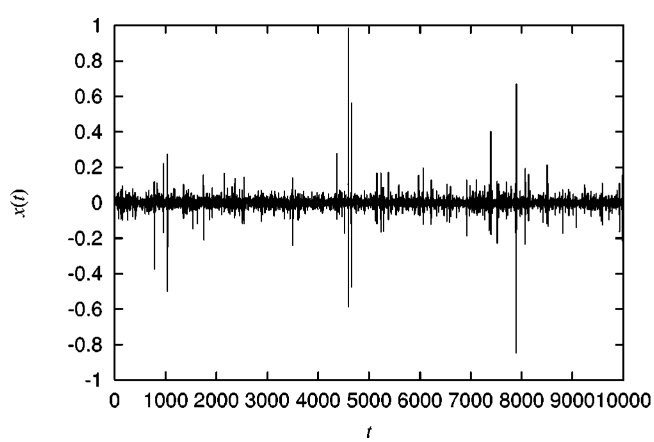

FIG. 5. Time series of $x(t)$ for $V=1.0$ in the case of the white Gaussian multiplicative noise.

III. THEORETICAL ANALYSIS

A. Renormalization procedure

Focusing on the relation that power law exponent β is inversely proportional to the autocorrelation time τ_c, we analyze the RMP theoretically. Our idea of derivation is based on a kind of renormalization procedure that has also

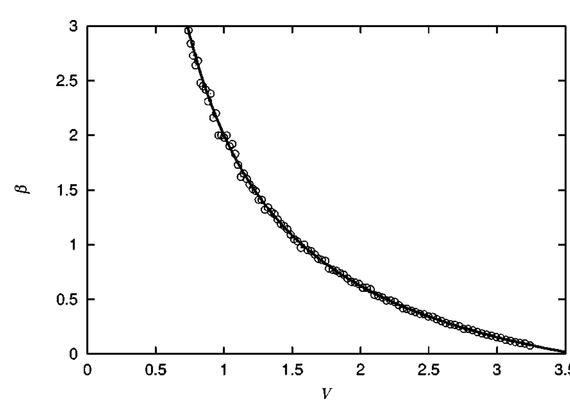

FIG. 7. The power law exponent β vs variance V of a white Gaussian multiplicative noise. Circles represent numerically estimated values and the curve gives the theoretical relation.

been argued by Nakao [16]. Substituting Eq. (1) recursively k times we have the renormalized representation from $t-k+1$ step to $t+1$ step

$$x(t+1)=\left\{\prod_{m=0}^{k-1} b(t-m)\right\}x(t-k+1)+\sum_{l=0}^{k-2}\left\{\prod_{m=0}^{l} b(t-m)\right\}f(t-l-1)+f(t). \tag{15}$$

Defining the renormalized multiplicative noise $B_k(N)$ and renormalized additive noise $F_k(N)$ as

$$B_k(N)=\prod_{m=Nk}^{(N-1)k-1} b(m-1), \tag{16a}$$

$$F_k(N)=\sum_{l=Nk}^{(N+1)k-2}\left\{\prod_{m=Nk}^{l} b(m-1)\right\}f(l-2)+f(kN-1), \tag{16b}$$

we obtain the renormalized equation with intervals k for $N=1,2,\ldots,$

$$x_k(N)=B_k(N)x_k(N-1)+F_k(N), \tag{17}$$

where $x_k(N)$ represents $x(kN-1)$.

B. Another representation of multiplicative noise

We introduce a stochastic variable λ_t defined by $b(t)=\exp(\lambda_t+i\theta_t)$, in order to approximate Eq. (17). Here θ_t represents a binary random variable taking the value of 0 and π with the same probability, which is required to represent the change of sign in $x_k(N)$ and $x_k(N-1)$. λ_t, of course, differs from that of $b(t)$. As we discuss in Sec. IV A we can make a reasonable assumption that λ_t is a stationary Gaussian noise having an average $\overline{\lambda}=\langle \ln|b|\rangle$ and its autocorrelation function κ_m decreases exponentially

$$\kappa_m=\langle\lambda_{n+m}\lambda_n\rangle-\langle\lambda_{n+m}\rangle\langle\lambda_n\rangle=D\exp(-m/\tau_c^*), \tag{18}$$

where D is a coefficient and τ_c^* is the autocorrelation time. The value of τ_c^* is shown to be nearly equal to τ_c by comparing the two autocorrelation functions, which we also show in Sec. IV.

The effect of θ_t does not play any role in the following calculation because θ_t is a factor that randomly reverses the sign of multiplicative noises and the effective amplitude is always unity. Approximating the renormalized multiplicative noise $B_k(N)$ by λ_t we have $B_k(N)=\exp(\Lambda_{k,M})$, where

$$\Lambda_{k,N}=\sum_{m=kN}^{(N-1)k-1}\lambda_{m-1}. \tag{19}$$

Since $\Lambda_{k,N}$ is stationary for N the autocorrelation function of $\Lambda_{k,N}$ on the renormalized level k, $A_{k,M}$ is given as

$$\begin{aligned}A_{k,M}&=\langle\Lambda_{k,M+N}\Lambda_{k,N}\rangle-\langle\Lambda_{k,M+N}\rangle\langle\Lambda_{k,N}\rangle\\&=\langle\Lambda_{k,M+1}\Lambda_{k,1}\rangle-\langle\Lambda_{k,M+1}\rangle\langle\Lambda_{k,1}\rangle\\&=\langle(\lambda_{Mk+1}+\cdots+\lambda_{(M-1)k})(\lambda_1+\cdots+\lambda_k)\rangle-k^2\overline{\lambda}^2\\&=\sum_{n=1}^{k-1}n\kappa_{(M+1)k-n}+k\kappa_{Mk}+\sum_{n=1}^{k-1}n\kappa_{(M-1)k-n}.\end{aligned}$$

Substituting Eq. (18) to the end term of the above equations we have

$$A_{k,M}=2D\frac{e^{-1/\tau_c^*}}{(1-e^{-1/\tau_c^*})^2}e^{-kM/\tau_c^*}[\cosh(k/\tau_c^*)-4], \tag{20}$$

which converges to a δ function in the limit of $k\to\infty$. As $\Lambda_{k,N}$ can be approximated by a white noise for large k, we may apply the equation determining the power law exponent Eq. (11) to the renormalized multiplicative noise Λ_k with large k.

On the other hand, it is easy to confirm that the autocorrelation function of the renormalized additive noise $F_k(N)$ is a white noise, actually the autocorrelation function of $F_k(N)$, $C_k(M)$, is calculated as

$$C_k(M)=\langle F_k(N)F_k(N+M)\rangle-\langle F_k(N)\rangle\langle F_k(N+M)\rangle=0. \tag{21}$$

Replacing b with e^{Λ_k} in Eq. (11) we have

$$\langle e^{\beta_k\Lambda_k}\rangle=1, \tag{22}$$

where β_k is the power law exponent of the renormalized variable $x_k(N)$. Rewriting Eq. (22) by an integral

$$\int P_\Lambda(\Lambda;k)e^{\beta_k\Lambda}d\Lambda=1, \tag{23}$$

where $P_\Lambda(\Lambda;k)$ represents the probability density of Λ_k in the renormalized level k. Moreover by making a transformation of the probability density involving Dirac's δ function from $P_\Lambda(\Lambda;k)$ to $q_k(\lambda_k,\ldots,\lambda_1)$,

$$P_\Lambda(\Lambda;k)=\int q_k(\lambda_k,\ldots,\lambda_1)\times\delta(\lambda_k+\cdots+\lambda_1-\Lambda)d\lambda_k\cdots d\lambda_1, \tag{24}$$

where $q_n(\lambda_1,\lambda_2,\ldots,\lambda_n)$ is a k-joint probability density of $\{\lambda\}$, which is represented by a k-dimensional Gaussian distribution

$$q_n(\lambda_1,\lambda_2,\ldots,\lambda_n)=\frac{1}{(2\pi)^{n/2}|R|^{1/2}}\exp\left(-\frac{1}{2}\sum_{i,j=1}^{n}R_{i,j}^{-1}(\lambda_i-\overline{\lambda})(\lambda_j-\overline{\lambda})\right), \tag{25}$$

where $R_{i,j}$ denotes the covariance matrix that is equal to $\kappa_{|i-j|}$, so that

$$R_{i,j} = \begin{bmatrix} D & De^{-1/\tau_c^*} & \cdots & D^{-(n-1)/\tau_c^*} \\ De^{-1/\tau_c^*} & D & & \vdots \\ \vdots & & & \\ De^{-(n-1)/\tau_c^*} & De^{-(n-2)/\tau_c^*} & \cdots & D \end{bmatrix}, \quad (26)$$

and $|R|$ means the determinant of $R_{i,j}$. From Eq. (23) and Eq. (24) we have

$$\int e^{\beta_k(\lambda_k+\cdots+\lambda_1)} q_k(\lambda_k,\ldots,\lambda_1)\, d\lambda_k \cdots d\lambda_1 = 1. \quad (27)$$

Substituting Eq. (25) for Eq. (27) we have

$$\exp\left[\beta_k k\overline{\lambda} + \frac{\beta_k^2}{2}\sum_{i,j=1}^{k} R_{i,j}\right] = 1, \quad (28)$$

so that

$$\beta_k k\overline{\lambda} + \frac{\beta_k^2}{2}\sum_{i,j=1}^{k} R_{i,j} = 0. \quad (29)$$

As a nonzero solution we get

$$\beta_k = -\frac{2k\overline{\lambda}}{\sum_{i,j=1}^{k} R_{i,j}}. \quad (30)$$

Calculating the denominator on the right-hand side using Eq. (18),

$$\beta_k = -\frac{2k\overline{\lambda}(1-e^{-1/\tau_c^*})^2}{D[k(1-e^{-2/\tau_c^*}) - 2e^{-1/\tau_c^*}(1-e^{-k/\tau_c^*})]}. \quad (31)$$

Finally in the limit of $k \to \infty$ we have

$$\beta_\infty = -\frac{2\overline{\lambda}(1-e^{-1/\tau_c^*})^2}{D(1-e^{-2/\tau_c^*})}. \quad (32)$$

It is easy to check the validity of Eq. (32) by computer simulations. We show plots of β_∞ as functions of τ_c^* estimating from the slope of a log-log plot of CDFs calculated numerically with fixing D and $\overline{\lambda}$ in Fig. 8. It is found that the numerical results fit with the theoretical curve. Expanding the exponentials in Eq. (32) over the powers, we take for large values of τ_c^* an approximation

$$\beta \approx -\frac{\overline{\lambda}}{D}\tau_c^{*-1}, \quad (33)$$

which means that β is inversely proportional to the autocorrelation time.

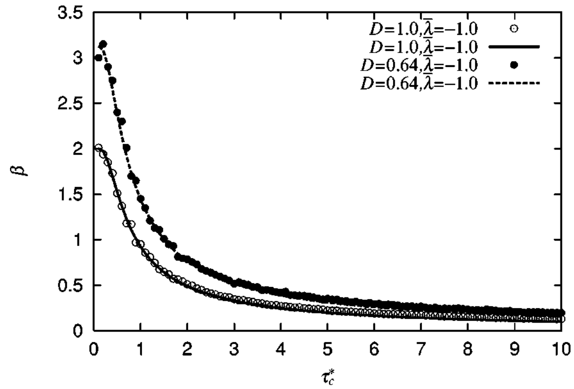

FIG. 8. Plots of β as functions of τ_c fixing D and $\overline{\lambda}$. Unfilled circles represent numerical results for $D=1.0$ and $\overline{\lambda}=-1.0$, and filled circles are for $D=0.64$ and $\overline{\lambda}=-1.0$. A solid curve represents the theoretical equation for $D=1.0$ and $\overline{\lambda}=-1.0$, and a dashed curve is for $D=0.64$ and $\overline{\lambda}=-1.0$.

IV. DISCUSSION

A. Relation between τ_c and τ_c^*

The autocorrelation time of λ_t, τ_c^*, in a theoretical calculation is not automatically identical to the autocorrelation time of b, τ_c, in a numerical simulation. In the following we verify $\tau_c \approx \tau_c^*$ by calculating a relation between $R(\tau)$ in Eq. (5) and $R^*(\tau)$ approximated by λ_t. The autocorrelation function $R^*(\tau)$ is expressed in terms of λ_t as

$$R^*(\tau) = \langle e^{\lambda_t} e^{\lambda_{t+\tau}}\rangle - \langle e^{\lambda_t}\rangle\langle e^{\lambda_{t+\tau}}\rangle. \quad (34)$$

From Eq. (26) one can calculate the first term and the second term on the right-hand side

$$\langle e^{\lambda_t} e^{\lambda_{t+\tau}}\rangle = \int e^{\lambda_t} e^{\lambda_{t+\tau}} q_n(\lambda_k,\ldots,\lambda_1)\, d\lambda_k \cdots d\lambda_1$$
$$= \exp[D\exp(-\tau/\tau_c^*)]e^{2\overline{\lambda}+D} \quad (35)$$

and

$$\langle e^{\lambda_t}\rangle = \langle e^{\lambda_{t+\tau}}\rangle = e^{2\overline{\lambda}+D}. \quad (36)$$

Therefore, we have

$$R^*(\tau) = \{\exp[D + D\exp(-\tau/\tau_c^*)] - e^{D}\}e^{2\overline{\lambda}}. \quad (37)$$

As shown in Fig. 9, $R^*(\tau)$ described by Eq. (37) can approximate Eq. (5) having the same value of τ_c as τ_c^*. Namely, τ_c^* is practically equal to τ_c. The agreement of the power law exponent of the probability density function between numerical simulation and that of the analytical derivation, in which higher-order moments of $\lambda_t = \ln|b(t)|$ are neglected, suggests that higher-order moments of a stochastic variable λ_t should not seriously affect the relation between the power law exponent of the probability density of the amplitude and the correlation time of multiplicative noise. However, the reason why the effects of the higher-order moments of multiplicative noise are negligible in the present system is an open question.

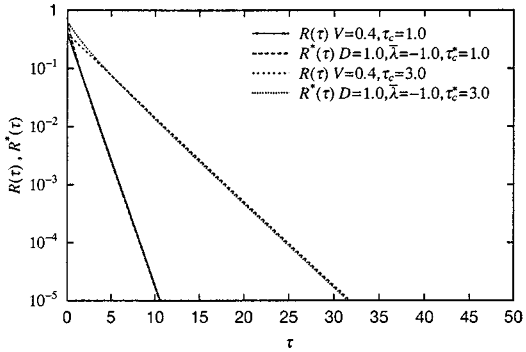

FIG. 9. Autocorrelation function of colored Gaussian noise $R(\tau)$ and autocorrelation function approximated by e^{λ_t}, $R^*(\tau)$. Notice that $R(\tau)$ is given appropriate approximation by R^* having the same value of τ_c^* as τ_c.

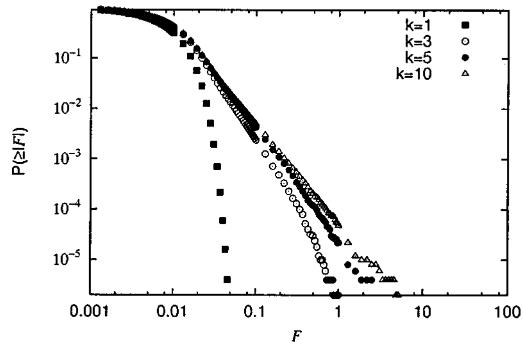

FIG. 10. Log-log plots of cumulative distributions of the renormalized additive noise for $k=1$, $k=3$, $k=5$, and $k=10$. We find clear power law tails for larger k, and their slopes are showing tendency to converge to a certain slope.

B. Applying the renormalization procedure to the white Gaussian case

Here, in order to confirm the validity of the renormalization procedure, we apply it to the RMP with a white multiplicative noise. Applying Eq. (11) to the renormalized multiplicative noise $B_k(N)$ we have

$$\langle |B_k|^{\beta_k} \rangle = \left\langle \left| \prod_{m=kN}^{k(N+1)-1} b(m-1) \right|^{\beta_k} \right\rangle . \tag{38}$$

Because $\{b(m)\}$ is a white noise, an average on the left-hand side can be calculated by using kth integrations such as

$$\begin{aligned} \left\langle \left| \prod_{m=kN}^{k(N+1)-1} b(m-1) \right|^{\beta_k} \right\rangle &= \int |b_1 b_2 \cdots b_k|^{\beta_k} W(b_1) W(b_2) \cdots W(b_k) \\ &\quad \times db_1 db_2 \cdots db_k \\ &= \langle |b|^{\beta_k} \rangle^k . \end{aligned} \tag{39}$$

We get $\langle b^{\beta_k} \rangle = 1$ for any renormalization level k from $\langle |B_k|^{\beta} \rangle = 1$. Actually it is obvious that the power law exponent β does not depend on the renormalization level k due to Eq. (11), and these results are consistent.

C. Distributions of renormalized additive noise

Contrary to the case without any amplification, the renormalized additive noise in the RMP is scaled by a power law distribution, not by a Gaussian. As shown in Fig. 10 it is found that its cumulative distribution converges rapidly to a stationary power law distribution, which has the same power law tail in the cumulative distribution of x. The reason for this power law can be attributed to the internal structure involving the term $\Pi_{m=Nk}^{l} b(m-1)$. Fortunately power law distributions of renormalized additive noise are not effective in estimation of the exponent β.

V. CONCLUSIONS

We have investigated numerically and analytically the power law exponent of the probability distribution for the amplitude of the discrete time model of Langevin systems with a stationary Gaussian colored multiplicative noise. We found that the exponent depends not only on the distribution of multiplicative noise but also on the correlation time of the multiplicative noise. We discovered by numerical simulation that the power law exponent is inversely proportional to the correlation time of the multiplicative noise for a large value of the correlation time and saturates to a finite value when it approaches zero. We showed that this experimental observation over a wide range of the correlation time can be verified analytically by applying a renormalizing procedure to the Langevin equation with multiplicative colored noise if the approximation that the logarithm of the colored Gaussian noise is also Gaussian is permitted. The reason that this assumption is permitted is an open problem for future study. The question is also a subject for future studies if the same relation holds between the correlation time and the power law exponent for the continuous time Langevin system with multiplicative colored noise. The results obtained in this paper would be useful by giving a universal relation between the power law exponent and the correlation time in the Langevin type of systems with multiplicative noise, which can be found in various systems as described in Sec. I.

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